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Boundary Behavior of Functions with Bounded Boundary Rotation

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1. INTRODUCTION

Let us denote by V_k the set of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1.1}$$

which are analytic in $U = \{z : |z| < 1\}$ and which satisfy

$$f'(z) = \exp \left\{ \frac{1}{\pi} \int_0^{2\pi} \log(1 - ze^{-it})^{-1} \, d\mu(t) \right\}, \tag{1.2}$$

where $\mu(t)$ is realvalued and of bounded variation on $[0, 2\pi]$ with

$$\int_0^{2\pi} d\mu(t) = 2\pi, \qquad \int_0^{2\pi} |d\mu(t)| \leqslant k\pi. \tag{1.3}$$

V_k may also be defined in terms of the concept of boundary rotation, introduced by Loewner [6]. If D is a schlicht domain with a C^1 boundary curve, the boundary rotation of D is defined as the total variation of the boundary tangent vector argument over a complete circuit. For more general domains, the rotation is defined by a limiting process. See [7] for a detailed description. Paatero [7] showed that $f(z)$ given by (1.1) belongs to V_k if and only if $f'(z) \neq 0$ in U and $f(U)$ is a domain with boundary rotation at most $k\pi$. He also showed that the boundary rotation of $f(U)$ is

$$\lim_{r \to 1} \int_0^{2\pi} \left| \operatorname{Re} \left\{ 1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} \right| d\theta. \tag{1.4}$$

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Note that it is geometrically obvious that $k \geqslant 2$. Also, for $k = 2$, V_k is the class of normalized convex functions. Paatero also showed that for $2 \leqslant k \leqslant 4$, V_k consists entirely of univalent functions.

In (1.2) it is clear that $\mu(t)$ is not uniquely determined by $f(z)$. However, if we require that $\mu(t)$ be normalized in the sense that

$$\mu(t) = \frac{(\mu(t+0) + \mu(t-0))}{2} \qquad \text{for all } t \in [0, 2\pi],$$

then an argument analogous to the proof of Lemma 1 in [9] shows that

$$A(t) = \lim_{r \to 1} \arg f'(re^{it}) \tag{1.5}$$

exists for all $t \in [0, 2\pi]$ and that $\mu(f; t) = t + A(t)$ satisfies conditions (1.2), (1.3), and is normalized. We call $\mu(f; t)$ *the measure associated with* $f(z)$. The purpose of this paper is to examine the relationship between the growth of $f(z)$ and the integrator $\mu(f; t)$.

Given such a $\mu(f; t)$ we shall write

$$\mu(f; t) = v(f; t) - \sigma(f; t)$$

for the canonical decomposition of $\mu(f; t)$ into the difference of nondecreasing functions. Specifically we have

$$v(f; t) = \tfrac{1}{2}\{\mu(f; t) + V_0{}^t(\mu)\},$$
$$\sigma(f; t) = \tfrac{1}{2}\{V_0{}^t(\mu) - \mu(f; t)\},$$

where $V_0{}^t(\mu)$ us the total variation of μ from 0 to t. Since $\mu(f; t)$ is normalized, elementary calculations show that $v(f; t)$ and $\sigma(f; t)$ are also normalized. It is also easy to verify that positive and negative jump discontinuities in $\mu(f; t)$ correspond respectively to jump discontinuities in $v(f; t)$ and $\sigma(f; t)$.

2. Boundary Behavior of V_k Functions

In [7] and [8] Paatero studied the class of functions which are both schlicht and of bounded boundary rotation. By a long and complicated argument he showed that such a function $f(z)$ is continuous in $|z| \leqslant 1$ except at a finite number of points z_j on $|z| = 1$, while for each z_j, $|f(z)| \to \infty$ as $z \to z_j$ in $|z| \leqslant 1$. In [3] Flett gave a shorter but still complicated proof of this same result. We give the following generalization of this theorem.

THEOREM 2.1. *Let $f(z) \in V_k$. Then there exist finite sets*

$$E_1 = \{e^{i\theta_1}, \ldots, e^{i\theta_n}\}$$

and $E_2 \subseteq E_1$ of points on $|z| = 1$, where $n \leqslant [k/2 + 1]$, with the following properties:

(i) *$f(z)$ has a continuous (finite-valued) extension to $|z| \leqslant 1$ except at points of E_2; for each $z_j \in E_2$, $|f(z)| \to \infty$ uniformly as $z \to z_j$ in $|z| \leqslant 1$.*

(ii) *$f(z)$ is absolutely continuous on any closed subset of $|z| = 1$ disjoint from E_1. On any such closed subset,*

$$\frac{df(e^{i\theta})}{d\theta} = ie^{i\theta} f'(e^{i\theta}) \quad \text{a.e.},$$

where $(df/d\theta)$ is the derivative with respect to values on $|z| = 1$, and

$$f'(e^{i\theta}) = \lim_{r \to 1} f'(re^{i\theta}).$$

Remarks.

1. Note that we no longer require that $f(z)$ be schlicht.

2. The square brackets $[\cdot]$ denote the greatest integer function. Although the estimate $n \leqslant [k/2 + 1]$ was undoubtedly known to both Paatero and Flett, it does not seem to have been explicitly stated.

3. When $f(z)$ is schlicht, Paatero also showed that every finite boundary arc is rectifiable. This may be used to show absolute continuity of $f(z)$ on $|z| = 1$. However, our proof yields absolute continuity for all $f(z) \in V_k$, and in a manner more conceptually simple than the proofs given by Paatero and Flett.

Proof. Let $\mu(f; t) = v(f; t) - \sigma(f; t)$ be the measure associated with $f(z)$ and let

$$E_1 = \{e^{i\theta} : v(f; \theta + 0) - v(f; \theta - 0) \geqslant \pi\}.$$

Since

$$\int_0^{2\pi} dv(f; t) \leqslant (k/2 + 1)\,\pi,$$

we see that E_1 is a finite set and $n \leqslant [k/2 + 1]$. We shall prove (ii) first, and then use parts of Flett's proof to prove (i). In order to prove (ii) it clearly suffices to show $f(z)$ has an absolutely continuous extension to ∂U in any sector $M = \{z : |z| \leqslant 1, a \leqslant \arg z \leqslant b\}$ not containing any points of E_1.

Let $M_1 = \{z : a - \gamma/2 \leqslant \arg z \leqslant b + \gamma/2\}$ where $\gamma > 0$ is chosen small enough so that $\{z : a - 2\gamma \leqslant \arg z \leqslant b + 2\gamma\} \cap E_1 = \phi$. Construct a convex domain $D \subseteq U$ such that:

1. $0 \in D$.
2. $\{z : | z | = 1, a - \gamma \leqslant \arg z \leqslant b + \gamma\} = \partial D \cap \partial U$.
3. The tangent vector to ∂D turns continuously.

Let $z = h(\xi)$ map $| \xi | < 1$ conformally onto D with $h(0) = 0$, $h'(0) > 0$. Then $| h(\xi)| \leqslant | \xi |$ by Schwarz's lemma.

We claim that there exists $\delta > 1$ such that $f'(h(\xi)) \in H_\delta$, where H_δ is the Hardy class. Let

$$c_1 = \frac{1}{2\pi} \int_0^{2\pi} dv(f; t),$$

$$c_2 = \frac{1}{2\pi} \int_0^{2\pi} d\sigma(f; t).$$

Then from the representation formula (1.2) (see also Theorem 3.1 in [2]),

$$f'(z) = \frac{(s_1(z)/z)^{c_1}}{(s_2(z)/z)^{c_2}}, \tag{2.1}$$

where $s_1(z)$ and $s_2(z)$ are normalized and starlike. Let $\beta\pi$ be the largest jump of $v(f; t)$ for $t \in (a - 2\gamma, b + 2\gamma)$. By our choice of a, b, and γ we see that $\beta < 1$. Also, by the manner in which $s_1(z)$ is constructed, we see that

$$s_1(z) = O \frac{1}{(1 - | z |)^{\beta/c_1}}$$

for $z \in D$.

Let $\delta > 1$ be chosen such that $\delta\beta < 1$. Then

$$\left| \frac{s_1(h(\xi))}{h(\xi)} \right| \leqslant \frac{A}{(1 - | h(\xi)|)^{\beta/c_1}} \leqslant \frac{A}{(1 - | \xi |)^{\beta/c_1}}.$$

But $s_1(h(\xi))$ is univalent in $| \xi | < 1$, so by a result of Spencer [5, p. 45] we have for any $\lambda > 0$ and $\rho \geqslant \rho_0$ that

$$\int_0^{2\pi} | s_1(h(\rho e^{i\varphi}))|^\lambda \, d\varphi \leqslant A(\rho_0, \lambda) + B(\lambda) \int_{\rho_0}^{\rho} \frac{M(t, s_1(h(te^{i\varphi})))^\lambda}{t} \, dt,$$

where $0 < \rho_0 < 1$ is fixed but arbitrary. Letting $\lambda = \delta c_1$, we have for $\rho \geqslant \rho_0$ that

$$\int_0^{2\pi} |f'(h(\rho e^{i\varphi}))|^{\delta} \, d\varphi \leqslant A(\rho_0, c_2) \int_0^{2\pi} |s_1(h(\rho e^{i\varphi}))|^{\delta c_1} \, d\varphi$$
$$\leqslant A(\rho_0, \delta, c_1) + B(\rho_0) \int_{\rho_0}^{\rho} \frac{dt}{(1-t)^{\delta\beta}}.$$

Since $\delta\beta < 1$, the right-hand side is bounded above as $\rho \to 1$. Thus $f'(h(\xi)) \in H_\delta$.

We now claim that $h'(\xi) \in H_{\delta'}$ where $1/\delta + 1/\delta' = 1$. Since we have chosen D to have as its boundary a smooth closed Jordan curve, this fact follows directly from [4, p. 425]. We could also prove it directly in the same manner we showed $f'(h(\xi)) \in H_\delta$.

Define $g(\xi) = f(h(\xi))$. We claim $g'(\xi) = f'(h(\xi))\, h'(\xi) \in H_1$. Let δ and δ' be as above. Since $f'(h(\xi)) \in H_\delta$ and $h'(\xi) \in H_{\delta'}$, the Hölder inequality shows that $g'(\xi) \in H_1$. Thus, by a theorem of F. Riesz [4, p. 409], $g(\xi)$ has an absolutely continuous extension to $|\xi| = 1$ and

$$\frac{d}{d\varphi} g(e^{i\varphi}) = ie^{i\varphi} g'(e^{i\varphi}) \quad \text{a.e. on } [0, 2\pi].$$

Now since $z = h(\xi)$ is a convex function, we know $\xi = h^{-1}(z)$ is continuous in the closure of D. Thus $f(z) = g(h^{-1}(z))$ is continuous in the closure of D, and in particular on M.

We now claim that $f(z)$ has an absolutely continuous extension to

$$M \cap \partial U = \{z \in \partial U : a \leqslant \arg z \leqslant b\}.$$

We know that $h(\xi)$ has a continuous extension to $|\xi| = 1$, and $h(\xi)$ maps $|\xi| = 1$ onto ∂D. Let

$$\Gamma = \left\{z \in \partial U : a - \frac{\gamma}{2} \leqslant \arg z \leqslant b + \frac{\gamma}{2}\right\}.$$

Then by condition 2 in the definition of D, the Schwarz reflection principle allows us to conclude that $h(\xi)$ is analytic in some open neighborhood of the closed set $h^{-1}(\Gamma)$. Clearly this analytic extension of $h(\xi)$ is also univalent. Thus, $h^{-1}(e^{i\theta})$ is welldefined and analytic on Γ, and hence absolutely continuous there. Also note that the univalence of $h^{-1}(e^{i\theta})$ implies that Γ is mapped onto $h^{-1}(\Gamma)$ in the direction of increasing (or decreasing) argument. From this and the fact that $g(\xi)$ is absolutely continuous on $h^{-1}(\Gamma)$, an elementary argument based on first principles shows that $f(z) = g(h^{-1}(z))$ is absolutely continuous on Γ, and hence on $M \cap \partial U$. This completes the proof of (ii).

We now begin the proof of (i). The proof we give does not depend on the fact that we are considering points of E_1. However, all this proof will yield is continuity on ∂U, and not absolute continuity. We first note the following fact: there exists some w_0 such that $f(z)$ is bounded away from w_0 if we restrict z to lie outside a sufficiently large compact subset $K \subseteq U$. This follows since $f(z) \in V_k$ is finitely valent [2].

With such a w_0, choose z_0 such that $f(z_0) = w_0$. Let

$$T(z) = \frac{z + z_0}{1 + \bar{z}_0 z}$$

and let

$$G(z) = \frac{f(T(z)) - w_0}{f'(z_0)(1 - |z_0|^2)}.$$

By a result of Robertson [10], $G(z) \in V_k$. Also $G(z)$ is bounded away from 0 for z outside K. Now $T(z)$ maps $\{z : |z| \leq 1\}$ conformally onto $\{z : |z| \leq 1\}$. Let $e^{i\theta_0} \in E_1$, and let $T(e^{i\theta_*}) = e^{i\theta_0}$. We now need the following technical lemma.

LEMMA 2.1. *With the above notation,* $\lim_{r\to 1} \arg G(re^{i\theta_*})$ *exists.*

Proof. Note that $\arg G(re^{i\theta_*})$ is well-defined (by continuation) for r sufficiently close to 1. Let $\{b_j\}_{j=1}^{v}$ be the zeros of $G(z)$ other than $z = 0$. [$G(z)$ is finitely valent.] Let

$$H(z) = \frac{G(z)}{z(z - b_1) \cdots (z - b_v)}.$$

Then $H(z) \neq 0$ in U since all zeros of $G(z)$ are simple. ($G'(z)$ never vanishes since $G(z) \in V_k$.) Thus $\arg H(re^{i\theta})$ is defined for all $re^{i\theta}$. Now

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{re^{i\theta} H'(re^{i\theta})}{H(re^{i\theta})} \right| d\theta = \int_0^{2\pi} |d \arg H(re^{i\theta})| \, d\theta$$

is the total variation of $\arg H(re^{i\theta})$ over $[0, 2\pi]$. Let

$$R = \max_{1 \leq j \leq v} \{|b_j|\}.$$

For $r > R$ we have

$$\arg H(re^{i\theta}) = \arg G(re^{i\theta}) + \sum_{j=1}^{v} \arg(re^{i\theta} - b_j)^{-1} + \arg re^{-i\theta}.$$

Thus for $r > R$,

$$\int_0^{2\pi} |d \arg H(re^{i\theta})|\, d\theta \leqslant \int_0^{2\pi} |d \arg G(re^{i\theta})|\, d\theta + (v+1)\, 2\pi.$$

Since $G(z) \neq 0$ for $|z| > R$, and since $G'(z) \neq 0$ in U, a theorem of Biernacki [1] gives (where $z = re^{i\theta}$)

$$\int_0^{2\pi} |d \arg G(re^{i\theta})|\, d\theta \leqslant \int_0^{2\pi} \left| \operatorname{Re} 1 + \frac{zG''(z)}{G'(z)} \right| d\theta.$$

Since $G(z) \in V_k$, combination of (1.4) with the above remarks shows that for $|z| > R$ we have, where $z = re^{i\theta}$, that

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{zH'(z)}{H(z)} \right| d\theta \leqslant k\pi + (v+1)\, 2\pi. \tag{2.2}$$

Since $H(z)$ never vanishes in U, (2.2) implies the existence of a constant $A(k)$ such that for $0 < r < 1$ we have

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{zH'(z)}{H(z)} \right| d\theta \leqslant A(k). \tag{2.3}$$

Now $H(z)$ is analytic and never vanishes in U, so $\arg H(z)$ is harmonic in U. Together with (2.3), this implies (see either [4, p. 387] or Theorem 2 in [3]) that there exists a periodic function $m(\theta)$ of bounded variation on $[0, 2\pi]$ such that $\arg H(re^{i\theta})$ is the Poisson integral of $m(\theta)$ and for all θ we have

$$\lim_{r \to 1} \arg H(re^{i\theta}) = m(\theta).$$

Recalling the definition of $H(z)$, we conclude that this implies existence of $\lim_{r \to 1} \arg G(re^{i\theta})$. This proves the lemma.

We are now able to use exactly the same methods as Flett to prove:

1. *$G(re^{i\theta_*})$ approaches a limit as $r \to 1$, infinite values being permitted.*

2. *$G(e^{i\theta})$ approaches a limit as $\theta \to \theta_* + 0$ and $\theta \to \theta_* - 0$, infinite values being permitted.*

We give a brief outline of the proof of 1. Suppose the desired limit did not exist. Since $\arg G(re^{i\theta_*})$ approaches a limit, $|G(re^{i\theta_*})|$ must not. By the mean-value theorem, we then find that $\arg G'(re^{i\theta_*})$ does not tend to a limit as $r \to 1$. This contradicts (1.5). See [3] for completely detailed proofs.

We are now able to prove (i). We know $\lim_{r\to 1} G(re^{i\theta_*})$ exists, we know [from (ii)] that $G(z)$ has a continuous extension to $|z| = 1$ in a neighborhood of $e^{i\theta_*}$ except perhaps at $e^{i\theta_*}$ itself, and we know that $\lim G(e^{i\theta})$ exists as $\theta \to \theta_* + 0$, $\theta \to \theta_* - 0$. Then a theorem of Lindelöf [4, p. 347] allows us to conclude that either $G(z)$ has a continuous finite-valued extension to $e^{i\theta_*}$, or $|G(z)| \to \infty$ uniformly as $z \to e^{i\theta_*}$, $|z| \leq 1$. Recalling the definition of $G(z)$, we see that (i) is completely proved. This finishes the proof of Theorem 2.1.

3. Order of a V_k Function on $|z| = 1$

One question which arises and about which Theorem 2.1 provides no information is the question concerning the rate at which a V_k function approaches ∞ as z tends toward a point in the set E_2. With this in mind, we give the following definition [5, p. 34].

Definition. Let $f(z)$ be analytic in U. The *order* of $f(z)$ at a point $\xi \in \partial U$ is defined by

$$\alpha(f; \xi) = \sup\{\delta > 0\text{: there exists a curve } \gamma \text{ ending at } \xi \text{ with } \liminf_{|z|\to 1}(1 - |z|)^{\delta} |f(z)| > 0, \text{ where } z \in \gamma.\}$$

If no such δ exists, we set $\alpha(f; \xi) = 0$. We now give some theorems concerning the order of $f(z)$ for $f(z) \in V_k$.

Theorem 3.1. *Let $f(z) \in V_k$, and let $\mu(f; t) = v(f; t) - \sigma(f; t)$ be the measure associated with $f(z)$. Let $\pi\alpha(\theta)$ be the jump of $v(f; t)$ at $e^{i\theta}$. Then*

$$\alpha(f; e^{i\theta}) = \max\{0, \alpha(\theta) - 1\}$$

and

$$\alpha(f'; e^{i\theta}) = \alpha(\theta).$$

Proof. If $\alpha(\theta) < 1$, then by Theorem 2.1 $f(z)$ is continuous and bounded in a neighborhood of $e^{i\theta}$, so $\alpha(f; e^{i\theta}) = 0 = \max\{0, \alpha(\theta) - 1\}$. Suppose now that $\alpha(\theta) \geq 1$. We first show $\alpha(f; e^{i\theta}) \leq \max\{0, \alpha(\theta) - 1\}$. It is sufficient to consider the case when $\alpha(f; e^{i\theta}) > 0$. Let $\epsilon > 0$ be given and choose $n > 0$ such that $v(\theta + n) - v(\theta - n) \leq \pi(\alpha(\theta) + \epsilon)$. Then from (1.2) it follows immediately that there exists $A(n)$ such that

$$|f'(re^{i\alpha})| \leq \frac{A(n)}{(1 - r)^{\alpha(\theta)+\epsilon}}$$

uniformly for $\varphi \in (\theta - n/2, \theta + n/2)$. This in turn shows that

$$|f(re^{i\varphi})| \leqslant \frac{A(n)}{(1-r)^{\alpha(\theta)-1+\epsilon}} \tag{3.1}$$

uniformly for $\varphi \in (\theta - n/2, \theta + n/2)$.

Let δ and γ be such that for $z \in \gamma$ we have

$$\liminf_{|z| \to 1} (1 - |z|)^{\delta} |f(z)| > 0. \tag{3.2}$$

Then eventually γ must remain in the sector $\theta - n/2 \leqslant \arg z \leqslant \theta + n/2$, so by (3.1) and (3.2) we have

$$\liminf_{|z| \to 1} A(n)(1-r)^{\delta - (\alpha(\theta)-1+\epsilon)} > 0.$$

Thus $\delta \leqslant \alpha(\theta) - 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\delta \leqslant \alpha(\theta) - 1$, and thus $\alpha(f; e^{i\theta}) \leqslant \alpha(\theta) - 1$.

We now show $\alpha(f; e^{i\theta}) \geqslant \max\{0, \alpha(\theta) - 1\}$. It is sufficient to assume $\alpha(\theta) > 1$. By (1.5) we know

$$m(\theta) = \lim_{r \to 1} \arg f'(re^{i\theta})$$

exists. Let c be chosen so that $\cos(\theta + m(\theta) + c) > 0$. Let $g(z) = e^{ic}f(z)$. Then

$$\begin{aligned} |f(re^{i\theta})| &= |g(re^{i\theta})| \\ &\geqslant \operatorname{Re} g(re^{i\theta}) \\ &= \operatorname{Re} \int_0^r e^{i\theta} g'(\rho e^{i\theta})\, d\rho \\ &= \int_0^r |f'(\rho e^{i\theta})| \cos\{\arg e^{i\theta} g'(\rho e^{i\theta})\}\, d\rho. \end{aligned} \tag{3.3}$$

Now $\arg\{e^{i\theta} g'(\rho e^{i\theta})\} \to \theta + c + m(\theta)$ as $\rho \to 1$. Thus choose $a > 0$ and $\rho_1 < 1$ such that $\rho \geqslant \rho_1$ implies

$$\cos\{\arg e^{i\theta} g'(\rho e^{i\theta})\} \geqslant a > 0.$$

By an argument similar to the proof of Lemma 1 in [9], we have that (1.2) implies that $\arg f'(re^{i\theta})$ is the Poisson integral of $\mu(f; \theta) - \theta$. Since this function is of bounded variation and since $\alpha(\theta) > 0$ [so $\sigma(f; t)$ is continuous at θ], we have [3, Lemma 2], as $\rho \to 1$,

$$\frac{\log |f'(\rho e^{i\theta})|}{\log \dfrac{1}{1-\rho}} \to \alpha(\theta). \tag{3.4}$$

Therefore, given $o < \epsilon < \alpha(\theta) - 1$ we combine (3.3) and (3.4) to find

$$\liminf_{r\to 1}(1 - r)^{\alpha(\theta)-1-\epsilon} | g(re^{i\theta})| \geqslant a/(\alpha(\theta) - 1 - \epsilon) > 0.$$

This implies

$$\alpha(f; e^{i\theta}) \geqslant \alpha(\theta) - 1.$$

In order to show $\alpha(f'; e^{i\theta}) = \alpha(\theta)$, we use the same general line of reasoning as above. This completes the proof of Theorem 3.1.

COROLLARY 3.2. *Let $f(z) \in V_k$. Then $\{e^{i\theta} : \alpha(f; e^{i\theta}) > 0\}$ is finite, and $\Sigma\alpha(f; e^{i\theta}) \leqslant k/2$. If n is the number of values of θ where $\alpha(f; e^{i\theta}) > 0$, then $n \leqslant [k/2 + 1]$; if k is an even integer, then $n < [k/2 + 1]$. If m is the number of points on $|z| = 1$ where $f(z) \to \infty$, then $m \leqslant [k/2 + 1]$.*

Remark. Note that if $k < 4$, the modulus of $f(z) \in V_k$ may become infinite at at most two distinct points of $|z| = 1$. This is wellknown for $k = 2$ (convex functions), and is best possible since we may map U onto an infinite strip.

Proof. The corollary follows immediately from Theorem 2.1 and Theorem 3.1.

COROLLARY 3.3. *Let $f(z) \in V_k$. Then*

$$\begin{aligned}\alpha(f; e^{i\theta}) &= \max\{0, \alpha(f'; e^{i\theta}) - 1\},\\ &= \max\{0, \alpha(\theta) - 1\}.\end{aligned}$$

Proof. Theorem 3.1.

Note that Corollary 3.3 is false for analytic functions in general, as is shown by the example

$$f(z) = (1 - z)^6 \exp\{-(1 + z)/(1 - z)\}.$$

It follows directly from the definition of order that $\alpha(f; 1) = 0$. However, if we let $c > 0$, $0 < \delta < 1$, and $h(r) > 0$ be given where $h(r) \to 0$ as $r \to 1$, we may define a curve $\gamma = \gamma(re^{i\theta})$ in U by the condition

$$\sin^2 \theta/2 = \frac{(1 - r)^\delta}{4r(c + h(r))} - \frac{(1 - r)^2}{4r}.$$

It then follows that $\gamma(re^{i\theta}) \to 1$ as $|z| \to 1$, and simple computations show

$$\liminf_{r\to 1}(1-r)^{2\delta}|f'(z)| \geqslant 2c^2 > 0 \qquad (z \in \gamma).$$

Thus $\alpha(f'; 1) \geqslant 2\delta$. Since $\delta < 1$ is arbitrary, we have $\alpha(f'; 1) \geqslant 2$, although $\alpha(f; 1) = 0$.

Since $\alpha(f; e^{i\theta})$ measures the rate at which $f(z) \in V_k$ becomes unbounded as $z \to e^{i\theta}$, one might expect some relation between $\max_\theta \alpha(f; e^{i\theta})$ and the growth rate of $M(r, f)$. Such a relation is easy to find. We have

THEOREM 3.4. *Let $f(z) \in V_k$, let $\mu(f; t)$ be the measure associated with $f(z)$, and let $\alpha\pi$ be the largest nonnegative jump of $\mu(f; t)$. Then*

$$\alpha = \max_\theta\{\alpha(f'; e^{i\theta})\} = \lim_{r\to 1}\frac{\log M(r, f')}{\log \dfrac{1}{1-r}} \tag{3.5}$$

and

$$\max\{\alpha - 1, 0\} = \max_\theta\{\alpha(f; e^{i\theta})\} = \lim_{r\to 1}\frac{\log M(r, f)}{\log \dfrac{1}{1-r}}. \tag{3.6}$$

Proof. In view of Theorem 3.1, all we must show is that the limits in (3.5) and (3.6) exist and equal α and $\max\{\alpha - 1, 0\}$ respectively. We prove (3.5) first. Choose θ such that $\alpha(\theta) = \alpha$. From (3.4) we see that given $\epsilon > 0$, there exists $r(\epsilon) < 1$ such that

$$M(r, f') \geqslant \frac{1}{(1-r)^{\alpha-\epsilon}}$$

for $r \geqslant r(\epsilon)$. This implies that

$$\liminf_{r\to 1}\frac{\log M(r, f')}{\log \dfrac{1}{1-r}} \geqslant \alpha. \tag{3.7}$$

For each $r < 1$, choose z_r such that $M(r, f') = |f'(z_r)|$. Then from (2.1) we have

$$\frac{\log M(r, f')}{\log \dfrac{1}{1-r}} \leqslant c_1\frac{\log M(r, s_1)}{\log \dfrac{1}{1-r}} - c_1\frac{\log r}{\log \dfrac{1}{1-r}} + c_2\frac{\log|r/s_2(z_r)|}{\log \dfrac{1}{1-r}}, \tag{3.8}$$

where $s_1(z)$ and $s_2(z)$ are starlike functions. Using an elementary distortion theorem [5, p. 4], we see that

$$\frac{\log | r/s_2(z_r) |}{\log \frac{1}{1-r}} \leqslant \frac{2 \log(1+r)}{\log \frac{1}{1-r}}. \tag{3.9}$$

Also, from Theorem 1 in [9] we have

$$\lim_{r\to 1} \frac{\log M(r, s_1)}{\log \frac{1}{1-r}} = \beta, \tag{3.10}$$

where β is the largest jump of the measure corresponding to $s_1(z)$. But by construction this measure is $v(t)/c_1$, so $\beta = \alpha/c_1$. Combining this with (3.8), (3.9), and (3.10), we see that

$$\limsup_{r\to 1} \frac{\log M(r, f')}{\log \frac{1}{1-r}} \leqslant \alpha. \tag{3.11}$$

Combination of (3.7) and (3.11) proves (3.5).

We now prove (3.6). If $\alpha < 1$, we see from the definition of E_1 in Theorem 2.1 that $f(z)$ is bounded in $| z | < 1$, so (3.6) follows trivially. Suppose now $\alpha \geqslant 1$. Since

$$f(z) = \int_0^z f'(\xi)\, d\xi,$$

where we integrate along a radius, we have

$$M(r, f) \leqslant \int_0^r M(\rho, f')\, d\rho. \tag{3.12}$$

If we now combine (3.5) and (3.12), we find that

$$\limsup_{r\to 1} \frac{\log M(r, f)}{\log \frac{1}{1-r}} \leqslant \alpha - 1. \tag{3.13}$$

Next note that if $\alpha = 1$, (3.13) implies (3.6), so we assume $\alpha > 1$. Let $r < 1$

be given and choose $z = re^{ix}$ such that $M(r, f') = |f'(z)|$. From the Cauchy integral formula we have

$$M(r, f') = |f'(z)| = \left| \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2} d\xi \right| \\ \leqslant \frac{2}{1 - r} M((1 + r)/2, f), \tag{3.14}$$

where C is the circle with radius $(1 - r)/2$ and center z.

Suppose now that

$$\liminf_{r \to 1} \frac{\log M(r, f)}{\log \dfrac{1}{1 - r}} = b < \alpha - 1. \tag{3.15}$$

Choose a sequence $t_n \nearrow 1$ with

$$\frac{\log M(t_n, f)}{\log \dfrac{1}{1 - t_n}} = b + \epsilon_n,$$

where $\epsilon_n > 0$ and $\epsilon_n \to 0$. Define $r_n < 1$ by $(1 + r_n)/2 = t_n$. Then from (3.14) we see that

$$M(r_n, f') \leqslant \frac{2}{1 - r_n} M(t_n, f) = \frac{2}{(1 - r_n)(1 - t_n)^{b + \epsilon_n}} \\ = \frac{2^{b+1+\epsilon_n}}{(1 - r_n)^{b+1+\epsilon_n}}.$$

This implies

$$\limsup_{n \to \infty} \frac{\log M(r_n, f')}{\log \dfrac{1}{1 - r_n}} \leqslant b + 1 < \alpha,$$

which contradicts (3.5). Thus (3.15) is false. Combining this with (3.13), we see that (3.6) is proved.

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